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STABILIZATION OF LINEAR STOCHASTIC SYSTEMS WITH STATE AND CONTROL DEPENDENT PERTURBATIONS

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Investigation of the stabilizability in the mean square of a linear system with several perturbations in the object and the control channel is reduced to the analysis of a simpler system with a smaller number of perturbations. Necessary and sufficient conditions of stabilizability are obtained, and a procedure for determining the stabilizing control is indicated. Effects of the perturbation pattern on the stabilizability of a system is investigated, and it is shown how such investigation can be simplified depending on the perturbation pattern.

1. Statement of the problem. The behavior of systems whose random perturbations depend on their state and control is often described by stochastic differential equations

$$x^{\cdot} = Ax + Bu + \sum_{r=1}^{k} \sigma_r x \xi_r^{\cdot} + \sum_{r=1}^{l} \psi_r u \eta_r^{\cdot} \tag{1.1}$$

Here x is the n-dimensional phase-coordinate vector, u is the m-dimensional control, A and σ_r are constant $n \times n$-matrices, $\xi_r^{\cdot}(t)$ $(r = 1, 2, \ldots, k)$ are perturbations in the object, $\eta_r^{\cdot}(t)$ $(r = 1, 2, \ldots, l)$ are perturbations in the control channel; all $\xi_r(t)$ and $\eta_r(t)$ are in the aggregate independent standard Wiener processes.

The questions of stability and stabilization of similar systems have been examined in many papers (see [1—12], for instance). In the present paper the dependence of random perturbations on the state and the control differs from (1.1), and is defined by the system $S_{k,l}$

$$S_{k,l}: x^{\cdot} = Ax + Bu + \sum_{r=1}^{k} \varphi_r \sqrt{x^* Q_r x}\, \xi_r^{\cdot} + \sum_{r=1}^{l} \vartheta_r \sqrt{u^* P_r u}\, \eta_r^{\cdot} \tag{1.2}$$

Here x, u, A, B, ξ_r and η_r are the same as in (1.1), φ_r and ϑ_r are constant n-vectors, Q_r and P_r are constant symmetric nonnegative definite $(Q_r \geqslant 0,\ P_r \geqslant 0)$ matrices of dimensions $n \times n$ and $m \times m$, respectively.

While in (1.1) the perturbation depends on the phase space point of the system in (1.2) it rather depends only on how far this point is from the coordinate origin, i.e., the dependence of the perturbations on the system's state in (1.2) is less fine than in (1.1). It turns out that the investigation of system (1.2) is simpler in many respects. The perturbations in system (1.1) are called perturbations of the first type, and those in (1.2), of the second type.

The necessary and sufficient conditions for the mean-square stability can be obtained for both systems by general methods connected either with the investigation of the corresponding systems for the second moments or with the Liapunov function. Stabilizability, however, is equivalent to the solvability of the Bellman equation for the corresponding stochastic optimization problem. It is impossible to regard these general methods as being sufficiently effective. For a very narrow but important class of systems with perturbations of the first type, more effective criteria of stability and stabilizability have been obtained [3, 4, 9, 11, 12], reducing the problem to the investigation of certain properties of corresponding deterministic systems. However, such a reduction is not possible in more general cases. In [10] the investigation of the stabilizability of a system with perturbation of the first type acting both in the object as well as in a scalar control channel was successfully reduced to the solving of a certain optimization problem for a system with perturbations in the object but without them in the control channel. Such an approach is systematically developed below.

2. Sequential stabilization procedure. In connection with system $S_{k,l}$ we consider $k+l+1$ systems

$$S_{0,0},\ S_{1,0},\ \ldots,\ S_{k,0},\ S_{k,1},\ \ldots,\ S_{k,l} \tag{2.1}$$

Each system in (2.1) is obtained by the rejection of a certain number of perturbations from system $S_{k,l}$. We introduce the following set of stabilizing controls

$$U_{s,r} \overset{\Delta}{=} \{u = -Kx \mid u \text{ stabilizes system } S_{s,r}\}$$

Clearly, $U_{s+1,r} \subset U_{s,r}$ and $U_{s,r+1} \subset U_{s,r}$. Let us first investigate the stabilizability of systems $S_{s,0}$, $1 \leqslant s \leqslant k$, i.e., of systems without perturbations in the control channel.

Theorem 2.1. Let system $S_{s-1,0}$ be stabilizable ($U_{s-1,0} \neq \varnothing$). Then for system $S_{s,0}$ to be stabilizable ($U_{s,0} \neq \varnothing$) it is necessary and sufficient that the inequality

$$\inf_{u \in U_{s-1,0}} I_s(u) < 1, \quad I_s(u) \overset{\Delta}{=} M \int_0^\infty x^* Q_s x \, dt \tag{2.2}$$

where $x(t)$ is the solution of system $S_{s-1,0}$ with $x(0) = \varphi_s$, be satisfied. The controls $u \in U_{s-1,0}$ stabilizing system $S_{s,0}$ are those and only those for which $I_s(u) < 1$.

Proof. Necessity. Let system $S_{s,0}$ with control $u = -Kx$ be stable ($u \in U_{s,0}$). Then for every positive definite matrix G ($G > 0$) we can find a matrix $M > 0$ satisfying the Liapunov equation

$$\Lambda_u(M) + \varphi_s^* M \varphi_s Q_s = -G \tag{2.3}$$

$$\Lambda_u(M) \overset{\Delta}{=} (A - BK)^* M + M(A - BK) + \sum_{r=1}^{s-1} \varphi_r^* M \varphi_r Q_r$$

Since $u \in U_{s-1,0}$, we can prove the existence of the inverse negative operator Λ_u^{-1}. Applying it to both sides of Eq. (2.3), we obtain

$$M + \varphi_s^* M \varphi_s \Lambda_u^{-1}(Q_s) = T_u(M) = -\Lambda_u^{-1}(G) > 0 \tag{2.4}$$

$$T_u(M) \overset{\Delta}{=} M - \Pi_u(M)$$

$$\Pi_u(M) \overset{\Delta}{=} -\varphi_s{}^* M \varphi_s \Lambda_u{}^{-1}(Q_s)$$

Operator Π_u is positive and, according to (2.4), satisfies the inequality $\Pi_u(M) < M$. Consequently, by Theorem 5.6 in [13] its single eigenvalue $\lambda = \varphi_s{}^* \Lambda_u{}^{-1}(Q_s)\varphi_s$ satisfies the inequality $\lambda < 1$ and, by Itô's theorem $I_s(u) < 1$.

S u f f i c i e n c y. Let the inequality $I_s(u) < 1$ be satisfied for some control $u \in U_{s-1,0}$. This means that the single eigenvalue $\lambda = I_s(u)$ of operator Π_u lies inside the unit circle. Hence it follows that operator T_u has an inverse, where $T_u^{-1}(C) = \Pi_u{}^0(C) + \Pi_u{}^1(C) + \ldots$, i.e., operator T_u^{-1} is positive. This signifies that for $G > 0$ the matrix $M \overset{\Delta}{=} T_u^{-1}[-\Lambda_u{}^{-1}(G)] > 0$ satisfies the equation

$$M - \Pi_u(M) = -\Lambda_u{}^{-1}(G)$$

Hence, because Eqs. (2.3) and (2.4) are equivalent, it follows that the matrix $M > 0$ satisfies Eq. (2.3), i.e., system $S_{s,0}$ with control u is stable. Consequently, every control $u \in U_{s-1,0}$ for which $I_s(u) < 1$ stabilizes system $S_{s,0}$.

Thus, in the investigation of the stabilizability of system $S_{k,0}$ a sequential procedure arises, at whose s-th step $(s = 1, 2, \ldots, k)$ an optimization problem is solved and inequality (2.2) verified. If this inequality is satisfied then system $S_{s,0}$ is stabilizable and we must pass on to the next stage, i.e., ascertain the stabilizability of system $S_{s+1,0}$. If this inequality is not satisfied, then system $S_{s,0}$ is not stabilizable, and, hence, system $S_{k,0}$ is not stabilizable. We go on to investigate the stabilizability of systems $S_{k,s}$ with $1 \leqslant s \leqslant l$, i.e., of systems with perturbations also in the control channel.

T h e o r e m 2. 2. Let system $S_{k,s-1}$ be stabilizable $(U_{k,s-1} \neq \varnothing)$. Then for the stabilizability of system $S_{k,s}$ $(U_{k,s} \neq \varnothing)$ it is necessary and sufficient that the inequality

$$\inf_{u \in U_{k,\, s-1}} J_s(u) < 1, \quad J_s(u) \overset{\Delta}{=} M \int_0^\infty u^* P_s u \, dt$$

where $x(t)$ is the solution of system $S_{k,s-1}$ with $x(0) = \vartheta_s$, be satisfied. Those and only those controls $u \in U_{k,s-1}$ for which $J_s(u) < 1$ stabilize system $S_{k,s}$.

The proof is analogous to that of Theorem 2.1. The procedure of sequential investigation of stabilizability, constructed on the basis of Theorem 2.1, can be extended in obvious fashion to systems with perturbations also in the control channel, thanks to Theorem 2.2. The realization of the whole procedure reduces to the sequential solving of $k + l$ optimal problems. At first these problems are solved for systems $S_{s-1,0}$ with criteria $I_s(u)$ $(s = 1, 2, \ldots, k)$, and next for systems $S_{k,s-1}$ with criteria $J_s(u)$ $(s = 1, 2, \ldots, l)$. Such problems are of independent interest. They have already been analyzed in the deterministic case (see [14], for instance).

N o t e. The sequence in which the perturbations are introduced can be arbitrary. Then at each step of the procedure it is necessary to solve an optimization problem, choosing functional $I(u)$ or $J(u)$ depending on whether the perturbation is introduced the object or in the control channel.

3. Solution of degenerate optimization problems and a stabilizability criterion in limit form. We consider stochastic systems $S_{p+1,q}$ and $S_{p,q+1}$. According to the Note in Sect. 2 the determination of the stabilizability of systems $S_{p+1,q}$ and $S_{p,q+1}$ is connected with the solving of the optimization problems

$$I(u) \to \inf_{u \in U}, \quad I(u) \stackrel{\Delta}{=} M \int_0^\infty x^* Q_{p+1} x \, dt \tag{3.1}$$

$$J(u) \to \inf_{u \in U}, \quad J(u) \stackrel{\Delta}{=} M \int_0^\infty u^* P_{q+1} u \, dt \tag{3.2}$$

$U \stackrel{\Delta}{=} \{u = -Kx \mid u$ stabilizes system $S_{p,q}\}$. Here $x(t)$ is the solution of system $S_{p,q}$. Problems (3.1) and (3.2) with degenerate quadratic criteria are the limiting cases of the corresponding nondegenerate problems of optimal stabilization.

We begin with problem (3.1). The relation

$$\inf_{u \in U} I(u) = \lim_{\varepsilon \searrow 0} \min_u I_\varepsilon(u) \tag{3.3}$$

$$I_\varepsilon(u) \stackrel{\Delta}{=} M \int_0^\infty [x^*(Q_{p+1} + \varepsilon G)x + \varepsilon u^* R u]\, dt, \quad G > 0, \; R > 0, \; \varepsilon > 0$$

is valid. If system $S_{p,q}$ is stabilizable, then

$$\min_u I_\varepsilon(u) = I_\varepsilon(u_\varepsilon) = x^*(0) M_\varepsilon x(0) \tag{3.4}$$

$$u_\varepsilon(x) = -[\varepsilon R + H(M_\varepsilon)]^{-1} B^* M_\varepsilon x \tag{3.5}$$

$$H(M) \stackrel{\Delta}{=} \sum_{r=1}^{q} \vartheta_r^* M \vartheta_r P_r, \quad F(M) \stackrel{\Delta}{=} \sum_{r=1}^{p} \varphi_r^* M \varphi_r Q_r$$

$M_\varepsilon > 0$ is the unique solution of the equation

$$A^*M + MA + F(M) - MB[\varepsilon R + H(M)]^{-1} B^* M = -Q_{p+1} - \varepsilon G \tag{3.6}$$

It can be shown that M_ε decreases monotonically as $\varepsilon \searrow 0$. Therefore, the limit $M_0 = \lim_{\varepsilon \searrow 0} M_\varepsilon$ exists. Then from (3.3) and (3.4) we obtain

$$\inf_{u \in U} I(u) = x^*(0) M_0 x(0)$$

Theorem 3.1. Let system $S_{p,q}$ be stabilizable $(U \neq \varnothing)$. Then for system $S_{p+1,q}$ to be stabilizable it is necessary and sufficient that the inequality $\varphi_{p+1}^* M_0 \varphi_{p+1} < 1$ is satisfied. The control u_ε found from (3.5) stabilizes system $S_{p+1,q}$ as soon as $\varphi_{p+1}^* M_\varepsilon \varphi_{p+1} < 1$.

Problem (3.2) is solved analogously. To formulate the corresponding theorem we introduce the equation

$$u_\delta(x) = -[P_{q+1} + \delta R + H(D_\delta)]^{-1} B^* D_\delta x \tag{3.7}$$

where $D_\delta > 0$ is the solution of the equation

$$A^*D + DA + F(D) - DB[P_{q+1} + \delta R + H(D)]^{-1} B^* D = -\delta G \tag{3.8}$$

$$(G > 0, \ R > 0, \ \delta > 0)$$

we set $D_0 = \lim_{\delta \searrow 0} D_\delta$.

Theorem 3.2. Let system $S_{p,q}$ be stabilizable $(U \neq \varnothing)$. Then for system $S_{p,\,q+1}$ to be stabilizable it is necessary and sufficient that the inequality $\vartheta_{q+1}{}^* D_0 \vartheta_{q+1} < 1$ be satisfied. The control u_δ found from (3.7) stabilizes system $S_{p,\,q+1}$ as soon as $\vartheta_{q+1}{}^* D_\delta \vartheta_{q+1} < 1$.

Let us mention two possible methods for solving Eqs. (3.6) and (3.8). These methods are highly effective in the computational sense. The first method is based on the fact that solutions of algebraic Eqs. (3.6) and (3.8) can be obtained as the stationary solutions of corresponding differential equations (see [15, 16], for example). The second method, suggested in [10, 17], is interesting in that the optimal stabilization problem for a stochastic system reduces to the successive solving of optimal stabilization problems for the corresponding deterministic system.

We remark that by using the ideas in Sects. 2 and 3 we can obtain sufficient conditions for the stabilizability of systems with perturbations of the first type.

4. Systems in which each perturbation acts on only one equation. Let the perturbations in system (1.2) be such that each one acts only on one equation. This means that vectors φ_r and ϑ_r have each only one nonzero coordinate. We take this coordinate equal to unity. Then all the perturbations are separated into n classes; the i-th class consists of perturbations acting on the i-th control. The whole set of indices of perturbations in both the object and the control channel can be separated into classes in corresponding manner. We set

$$V_i = \{r \mid \varphi_r = e_i, \ 1 \leqslant r \leqslant k\}, \ W_i = \{r \mid \vartheta_r = e_i, \ 1 \leqslant r \leqslant l\}$$

where e_i is an n-dimensional vector whose i-th coordinate equals unity while the rest equal zero.

We now consider the $n + 1$ systems

$$S_0 : x^\cdot = Ax + Bu, \quad s = 0 \tag{4.1}$$

$$S_s\colon x^\cdot = Ax + Bu + \sum_{i=1}^{s} e_i \Big[\sum_{r \in V_i} \sqrt{x^* Q_r x}\, \xi_r^\cdot + \sum_{r \in W_i} \sqrt{u^* P_r u}\, \eta_r^\cdot \Big]$$

$$1 \leqslant s \leqslant n$$

We introduce the set of stabilizing controls $U_s = \{u = -Kx \mid u$ stabilizes system $S_s\}$.

Theorem 4.1. Let system S_{s-1} be stabilizable $(U_{s-1} \neq \varnothing)$. For system S_s to be stabilizable $(U_s \neq \varnothing)$ it is necessary and sufficient that the inequality

$$\inf_{u \in U_{s-1}} M \int_0^\infty [x^* G_s x + u^* R_s u]\, dt < 1$$

$$G_s = \sum_{r \in V_s} Q_r, \quad R_s = \sum_{r \in W_s} P_r$$

where $x(t)$ is the solution of system S_{s-1} with $x(0) = e_s$, be satisfied. Those and only those controls $u \in U_{s-1}$ for which

$$M \int_0^\infty [x^* G_s x + u^* R_s u]\, dt < 1$$

stabilize system S_s.

The proof is analogous to that of Theorem 2.1.

Now the stabilizability of system (1.2) (S_n) can be ascertained not in $k + l$ steps, as in the general case, but in n steps, since at each step the influence of all perturbations acting on one control is investigated at the same time. In addition, if $G_s > 0$ and $R_s > 0$, then at the s-th step this investigation is connected with the solving of a traditional optimal stabilization problem with a nondegenerate criterion.

5. Stabilizability of systems with arbitrary perturbations in the object. The constructive-algorithmic nature of the method proposed in Sects. 2 and 3 enables us to ascertain the stabilizability of a system only when we have concrete parameters. However, the general approach on which this method is based can also yield qualitative results.

Theorem 5.1. Let system $S_{0,0}$ be stabilizable and let Y be a linear subspace of phase space X. The following statements are equivalent.

1°. System $S_{k,0}$ is stabilizable for any perturbations acting on Y (i.e., $\varphi_r \in Y$, Q_r are arbitrary).

2°. The equality

$$\inf_{u \in U_{0,0}} \int_0^\infty x^*(t)\, x(t)\, dt = 0$$

is valid for the solution $x(t)$ of system $S_{0,0}$ with initial condition $x(0) \in Y$.

3°. $Y \subset \text{Range } B$ (Range B is the range of matrix B).

Proof. The equivalence of the first two statements can be checked by analogy with the proof of Theorem 2.1. To prove the equivalent of statements 2° and 3° we start from the relation

$$\inf_{u \in U_{0,0}} \int_0^\infty x^*(t)\, x(t)\, dt = x^*(0) M_0 x(0) \tag{5.1}$$

where $M_0 = \lim_{\varepsilon \searrow 0} M_\varepsilon$, M_ε is the unique solution of the equation

$$A^* M + MA - \varepsilon^{-2} MBB^* M = -E \tag{5.2}$$

(E is the unit matrix). From relation (5.1) we get that every space Y of initial values, for which statement 2° is valid, belongs to the space Y_0 of all solutions of the equation $y^* M_0 y = 0$ which, because of the symmetry of matrix M_0, is equivalent to the equation

$$M_0 y = 0 \tag{5.3}$$

Let us prove that $Y_0 = \text{Range } B$. Multiplying both sides of relation (5.2) from the left by y^* and from the right by y, we obtain

$$y^* A^* M y + y^* M A y - \varepsilon^{-2} y^* MBB^* M y = -y^* y$$

Hence from (5.3) it follows that for some $\varepsilon > 0$ the inequality

$$y^* M_\varepsilon BB^* M_\varepsilon y > 0 \tag{5.4}$$

is satisfied for every nonzero $y \in Y_0$. We set $Z \triangleq M_\varepsilon Y_0$. Then from (5.4) we get that the inequality $zBB^*z > 0$ is satisfied for every nonzero $z \in Z$; consequently, $r(B) \geqslant \dim Z$. Since $M_\varepsilon > 0$, we have $|M_\varepsilon| \neq 0$ and, thus, $\dim Z = \dim Y_0$. Therefore we obtain

$$r(B) \geqslant \dim Y_0 \tag{5.5}$$

Further, multiplying both sides of relation (5.2) by ε^2 and letting $\varepsilon \searrow 0$, we obtain $M_0BB^*M_0 = 0$, whence follows the equality $M_0B = 0$ which signifies that the columns of matrix B are the solutions of Eq. (5.3), i.e., $\mathrm{Range}\, B \subset Y_0$. Comparing this relation with inequality (5.5), we find that $Y_0 = \mathrm{Range}\, B$, whence the equivalence of statements 2° and 3° follows at once.

Corollary. For the stabilizability of system $S_{k,0}$ with any perturbations (i.e., for any φ, and Q_r) it is necessary and sufficient that matrix B be nonsingular and has the dimension $n \times n$.

An incorrect sufficient criterion for the stabilizability of systems with arbitrary perturbations of the first type in the object was presented in [10] (Theorem 3). As implied by the corollary, which is also valid for perturbations of the first type, complete controlability is insufficient for such stabilizability. This error was mentioned in [18]. We point out that the incorrect criterion in [10] has no bearing on its main contents and that all the rest of the material in [10] does not rely on this criterion.

6. Example. Consider the system

$$\begin{aligned} x_1^{\cdot} &= x_2 + \alpha \sqrt{x_1^2 + x_2^2}\,\xi_1^{\cdot}, \quad \alpha > 0 \\ x_2^{\cdot} &= u + \sqrt{g_1^2x_1^2 + g_2^2x_2^2}\,\xi_2^{\cdot} + \beta\,|u|\,\eta^{\cdot}, \quad g_1 > 0, \quad g_2 > 0, \quad \beta > 0 \end{aligned} \tag{6.1}$$

The corresponding determinate system

$$x_1^{\cdot} = x_2, \; x_2^{\cdot} = u \tag{6.2}$$

is stabilizable. We first investigate the stabilizability of the system

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = u + \sqrt{g_1^2x_1^2 + g_2^2x_2^2}\,\xi_2^{\cdot} + \beta\,|u|\,\eta^{\cdot} \tag{6.3}$$

For this, according to Theorem 4.1, we need to solve the optimization problem

$$\min_u \int_0^\infty [g_1^2x_1^2 + g_2^2x_2^2 + \beta^2u^2]\,dt$$

for system (6.2) with $x_1(0) = 0$, $x_2(0) = 1$. Note that in this case the corresponding Riccati algebraic equation

$$A^*M + MA - \beta^{-2}Mbb^*M = -G, \; G = \begin{Vmatrix} g_1^2 & 0 \\ 0 & g_2^2 \end{Vmatrix} \tag{6.4}$$

is easily solved and that matrix M has the elements

$$m_{11} = g_1\sqrt{2\beta g_1 + g_2^2}, \quad m_{12} = \beta g_1, \quad m_{22} = \beta\sqrt{2\beta g_1 + g_2^2} \tag{6.5}$$

Then for system (6.3) we obtain the stabilizability condition

$$m_{22} = \beta\sqrt{2\beta g_1 + g_2^2} < 1$$

now setting $\beta = 0.2$, $g_1 = g_2 = 0.1$, we ascertain the stabilizability of system (6.1).

For this, according to Theorem 2.1, for system (6.3) with $x_1(0) = \alpha$, $x_2(0) = 0$ we need to solve the degenerate optimization problem

$$\inf_{u \in U} M \int_0^\infty [x_1^2 + x_2^2]\, dt$$

where $U \triangleq \{u = -k_1 x_1 - k_2 x_2 \mid u$ stabilizes system (6.3)$\}$. According to Sect. 3 this problem is connected with solving the equation

$$A^*M + MA + \varphi^* M \varphi G - Mbb^*M \,/\, (\varepsilon + \vartheta^* M \vartheta) = -E \tag{6.6}$$

$$\varphi^* M \varphi = m_{22}, \quad \vartheta^* M \vartheta = 0.04 m_{22}, \quad E = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

The latter can be solved in various ways. In this case using formulas (6.5), it is easy to realize the iteration process

$$\begin{gathered} A^*M^{(s)} + M^{(s)}A - M^{(s)}bb^*M^{(s)} \,/\, (\varepsilon + \vartheta^* M^{(s-1)} \vartheta) = \\ -E - \varphi^* M^{(s-1)} \varphi G\, M^{(0)} = 0 \end{gathered} \tag{6.7}$$

Relation (6.7) with $\varepsilon > 0$ determines a sequence of matrices $M_\varepsilon^{(s)} > 0$ $(s = 1, 2, \ldots)$, which converges (see [17]) to a matrix $M_\varepsilon > 0$ satisfying Eq. (6.6). Thus, for any $\varepsilon > 0$ we can find M_ε and, then M_0. In the considered case the elements of matrix M_0, to within $0.5 \cdot 10^{-5}$, are: $m_{11}^\circ = 1.04125$, $m_{12}^\circ = 0.04165$, $m_{22}^\circ = 0.04335$. In accordance with Theorem 3.1 we obtain the condition

$$\alpha < 1 \,/\, \sqrt{m_{11}^\circ} = 0.97999$$

for the stabilizability of system (6.1) with $\beta = 0.2$, $g_1 = g_2 = 0.1$.

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